

# A second-order semi-Lagrangian particle FEM method for the incompressible Navier-Stokes equations

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# OUTLINE

- 
- ✓ Introduction
- ✓ Semi-Lagrangian approach
- ✓ SL-PFEM
- ✓ Verification and convergence analyses
- ✓ Taylor-Green vortex
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- ✓ Lid driven cavity flow
- ✓ 3D flow past a cylinder
- ✓ Ongoing and future work
- ✓ Acknowledgements
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## INTRODUCTION

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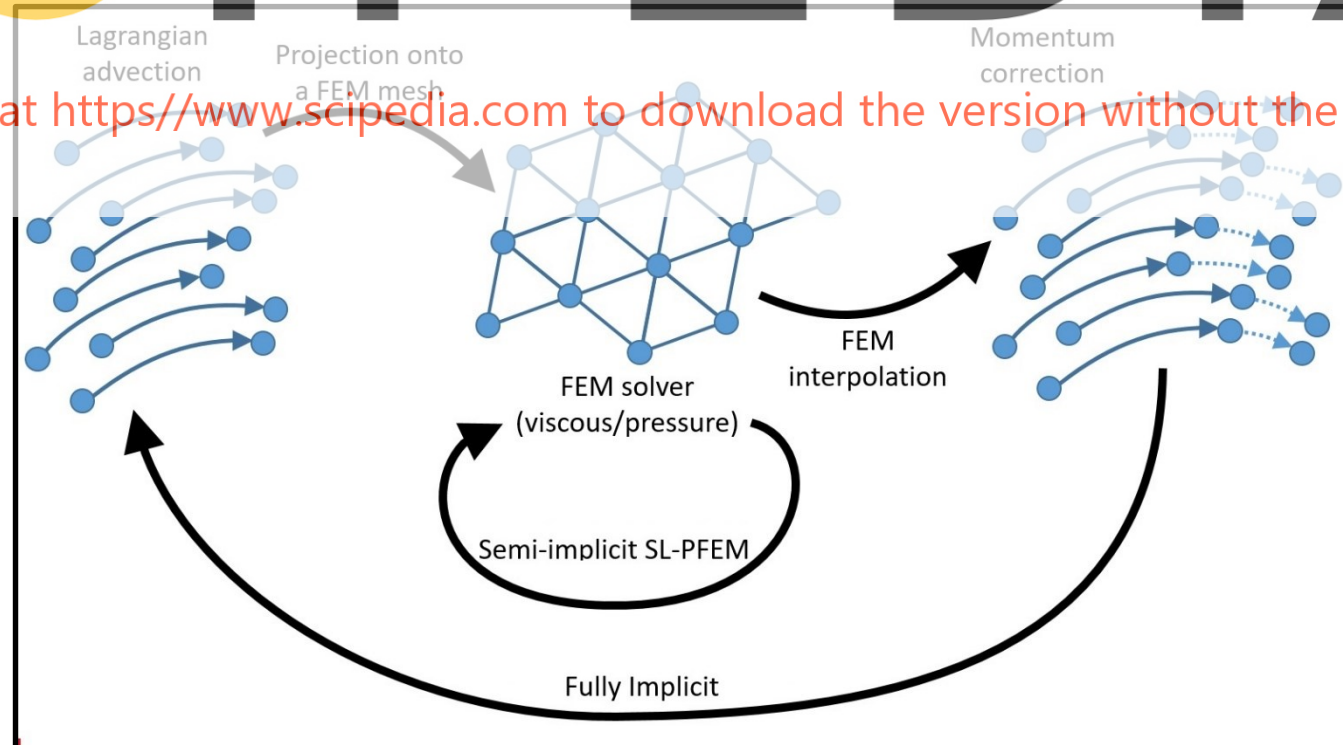
# SEMI-LAGRANGIAN APPROACH

## Concepts of the Semi-Lagrangian particle Finite Element Method (SL-PFEM)

First introduced by: S. R. Idelsohn, N. Nigro, A. Limache, E. Oñate: Large time-step explicit integration method for solving problems with dominant convection. Computer Methods in Applied Mechanics and Engineering 217-220, 168–185 (2012). DOI 10.1016/j.cma.2011.12.008.

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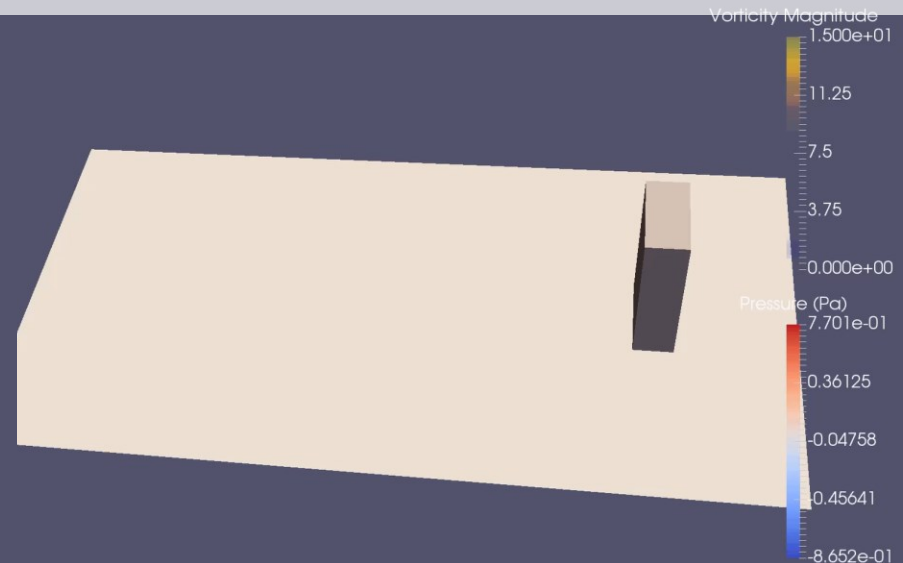
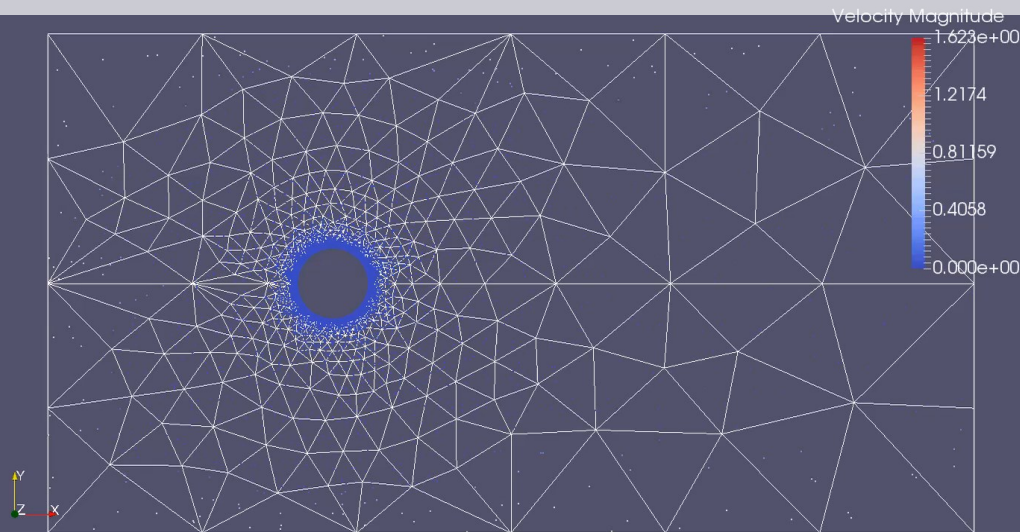
# SEMI-LAGRANGIAN APPROACH

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Velocity Magnitude

2.000e+00  
1.5012  
1.0023  
0.50347  
4.625e-03

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# SEMI-LAGRANGIAN APPROACH

## Integrating the particle's equation of motion

Let  $a(x, t)$  be an acceleration field and let  $\{\lambda\}$  be a set of particles each of them identified with a label  $\lambda$ .

Particle's equation of motion:

$$\begin{aligned} d_t U_\lambda(t) &= A_\lambda(t) = a(X_\lambda(t), t) \\ d_t X_\lambda(t) &= U_\lambda(t) \end{aligned}$$

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Velocity Verlet algorithm:

$$\begin{aligned} X_\lambda(t^{n+1}) &= X_\lambda(t^n) + \Delta t U_\lambda(t^n) + \frac{\Delta t^2}{2} A_\lambda(t^n) + O(\Delta t^3) \\ U_\lambda(t^{n+1}) &= U_\lambda(t^n) + \frac{\Delta t}{2} (A_\lambda(t^n) + A_\lambda(t^{n+1})) + O(\Delta t^3) \end{aligned}$$

## SL-PFEM for the incompressible Navier-Stokes equations

Let  $\mathbf{u}(\mathbf{x}, t)$  be a fluid velocity field and let's define the acceleration field:

$$\mathbf{a} = d_t \mathbf{u} = \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \left( \frac{p}{\rho} \right) + \nu \Delta \mathbf{u} + \mathbf{f}$$

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$$\begin{aligned} \mathbf{X}_\lambda^{n+1} &= \mathbf{X}_\lambda^n + \Delta t \mathbf{u}^n(\mathbf{X}_\lambda^n) + \frac{\Delta t}{2} \mathbf{a}^n(\mathbf{X}_\lambda^n) \\ \mathbf{U}_\lambda^{n+1} &= \underbrace{\mathbf{U}_\lambda^n + \frac{\Delta t}{2} \mathbf{a}^n(\mathbf{X}_\lambda^n)}_{\mathbf{U}_\lambda^{n+1/2}} + \frac{\Delta t}{2} \underbrace{\mathbf{a}^{n+1}(\mathbf{X}_\lambda^{n+1})}_{\text{Implicit}} \end{aligned}$$

$\mathbf{u}(\mathbf{X}_\lambda)$ ,  $\mathbf{a}(\mathbf{X}_\lambda)$ : Interpolated mesh velocity and acceleration

## SL-PFEM for the incompressible Navier-Stokes equations

Projection onto FEM mesh to solve  $\mathbf{a}^{n+1}$  on Eulerian description such that  $\mathbf{u}^{n+1}(\mathbf{x}) = \mathcal{P}_{\{\lambda\}}^{n+1} [\{\mathbf{U}_{\lambda}^{n+1}\}]$ :

$$\mathbf{U}_{\lambda}^{n+1} = \mathbf{U}_{\lambda}^{n+1/2} + \frac{\Delta t}{2} \mathbf{a}^{n+1}(\mathbf{X}_{\lambda}^{n+1})$$

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$$\mathbf{u}^{n+1}(\mathbf{x}) = \underbrace{\mathcal{P}_{\{\lambda\}}^{n+1} [\{\mathbf{U}_{\lambda}^{n+1/2}\}]}_{\mathbf{u}^{n+1/2}} + \frac{\Delta t}{2} \mathcal{P}_{\{\lambda\}}^{n+1} [\{\mathbf{a}^{n+1}(\mathbf{X}_{\lambda}^{n+1})\}]$$

Coherence condition:  $\mathbf{a}^{n+1}(\mathbf{x}) = \mathcal{P}_{\{\lambda\}}^{n+1} [\{\mathbf{a}^{n+1}(\mathbf{X}_{\lambda}^{n+1})\}]$

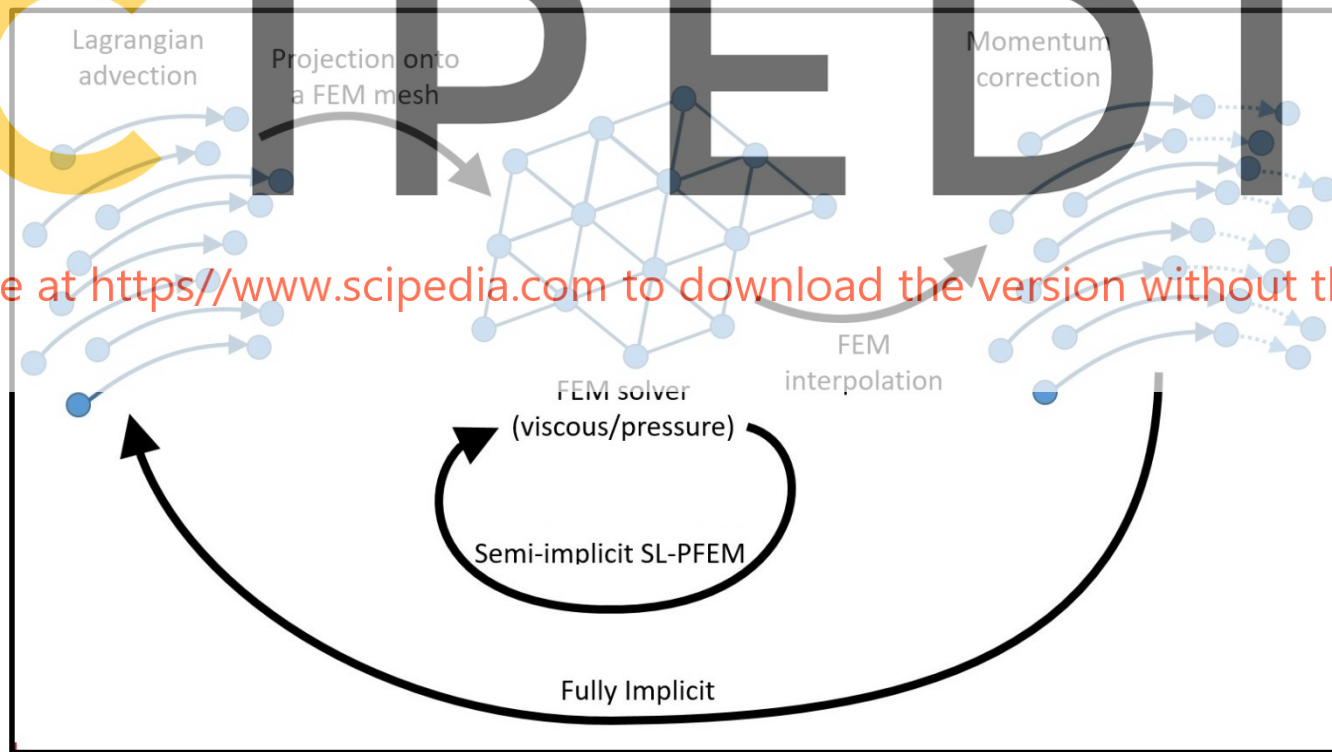


# SL-PFEM

SL-PFEM for the incompressible Navier-Stokes equations

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**Remark:** the coherence condition makes it unnecessary to iterate at the outer implicit loop.

## Projection

Minimization of the least square error (LSE).

$\{\Psi_\lambda\}$ : set of particles' values  $\xrightarrow{\text{Projection}} \mathcal{P}[\{\Psi_\lambda\}] = \{\psi_c^*\}$  projected nodal values

Interpolated-projected values on particles:  $\psi_h(\mathbf{X}_\lambda) = \sum_c N^c(\mathbf{X}_\lambda) \psi_c^*$

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Square error:  $\epsilon_\psi = \sum_\lambda (\psi_h(\mathbf{X}_\lambda) - \Psi_\lambda)^2$

$$\text{LSE: } \frac{\partial \epsilon_\psi}{\partial \psi_b^*} = 0 \rightarrow \sum_\lambda \left( \sum_c N^b(\mathbf{X}_\lambda) N^c(\mathbf{X}_\lambda) \psi_c^* \right) = \sum_\lambda N^b(\mathbf{X}_\lambda) \Psi_\lambda$$

Fulfils the coherence condition naturally

$$\Psi_\lambda = \psi_h(\mathbf{X}_\lambda) \rightarrow \epsilon_\psi = 0$$

# SL-PFEM

## Semi-Lagrangian approach for the incompressible Navier-Stokes equations

Equations in the Eulerian description:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n+1/2}}{\Delta t} = \underbrace{\frac{1}{2} \left( -\nabla \left( \frac{p^{n+1}}{\rho} \right) + \nu \Delta \mathbf{u}^{n+1} + \mathbf{f}^{n+1} \right)}_{\mathbf{a}^{n+1}}$$

$$\nabla \cdot \mathbf{u}^{n+1} = 0$$

Solved using FEM implicit scheme inspired in the fractional step method.

# VERIFICATION AND CONVERGENCE ANALYSES



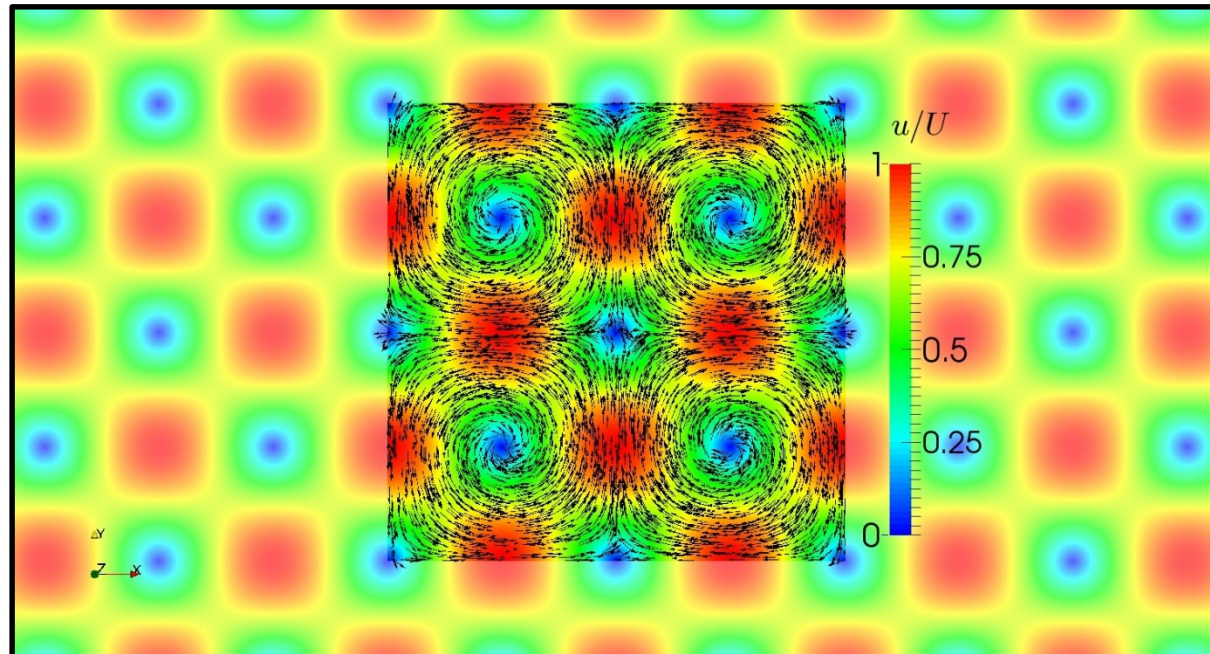
# TAYLOR-GREEN VORTEX

## Taylor-Green vortex solution

$$u_x(x, y, t) = -\sin(x) \cos(y) e^{-2\nu t}$$

$$u_y(x, y, t) = +\cos(x) \sin(y) e^{-2\nu t}$$

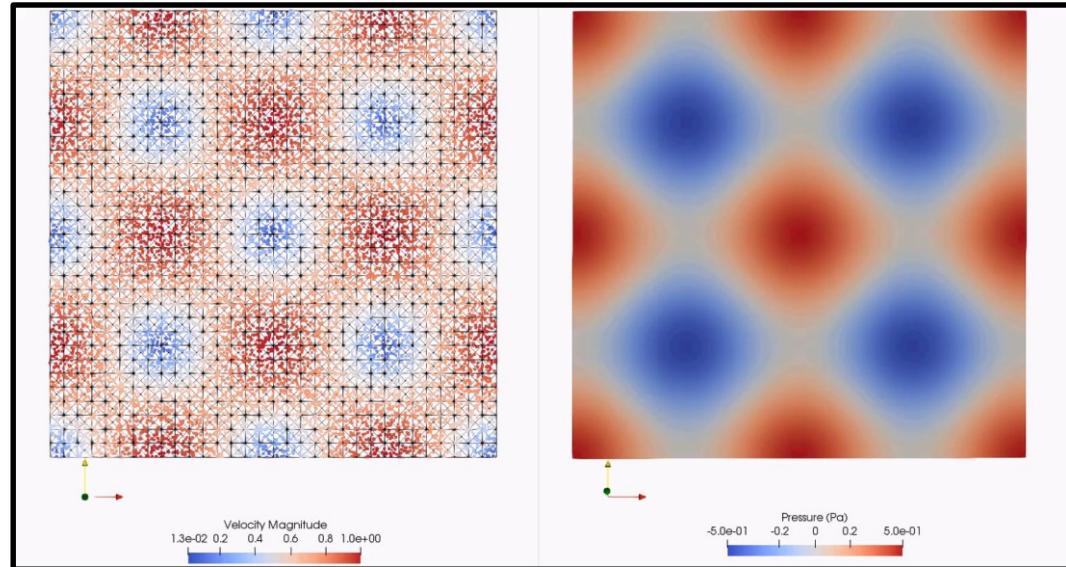
$$P(x, y, t) = \frac{1}{4} [\cos(2x) + \cos(2y)] e^{-4\nu t}$$



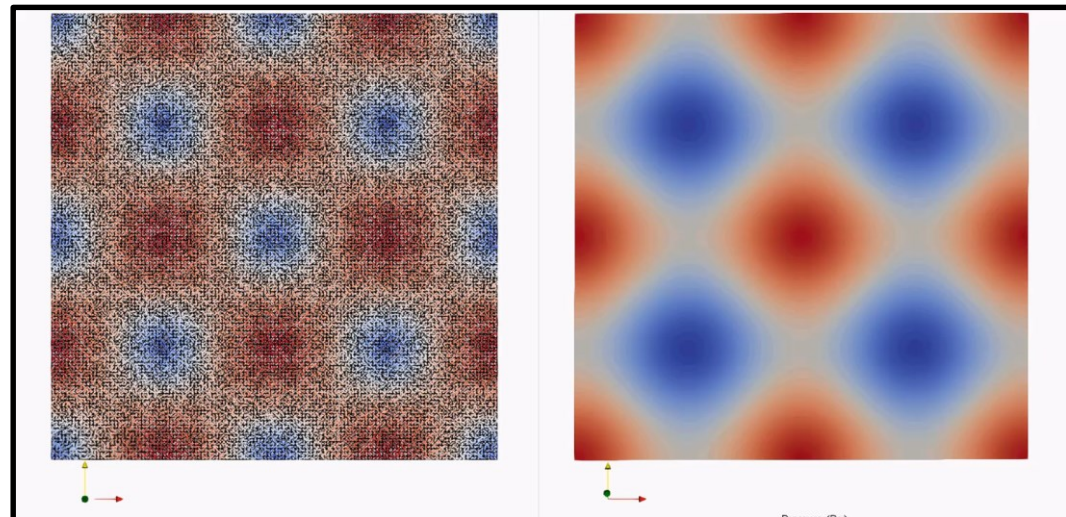


# TAYLOR-GREEN VORTEX

Case 32	
Number of elements	4096
Number of Nodes	2113
Number of Particles	12288
Courant number	2.54
Reynolds number	2000
Number of time steps	500
Simulation time	250 s



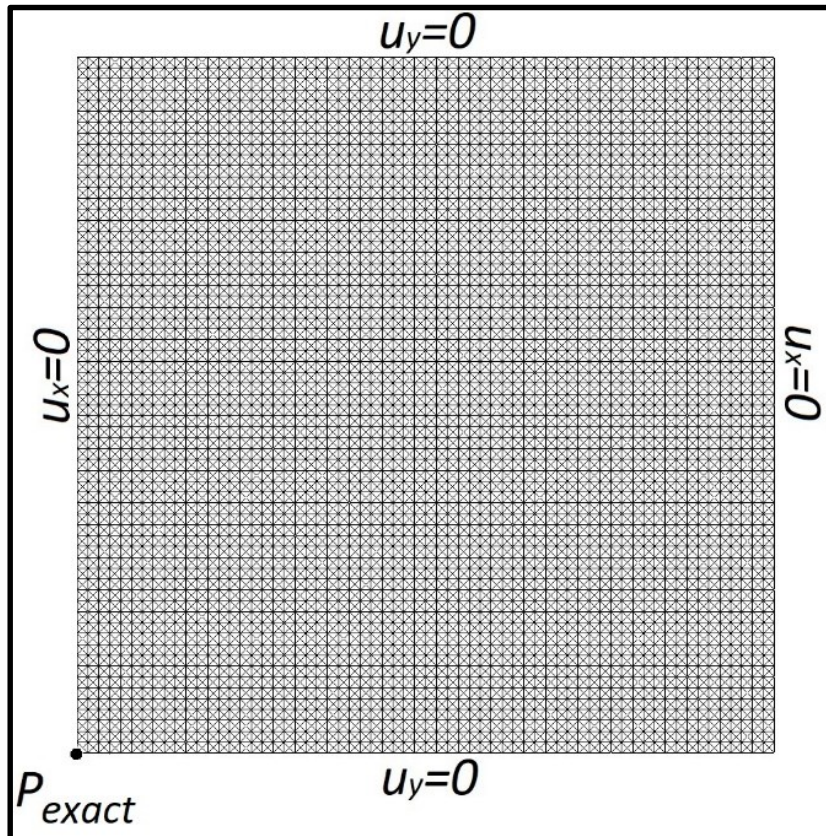
Case 128	
Number of elements	65536
Number of Nodes	33025
Number of Particles	196608
Courant number	10.1
Reynolds number	2000
Number of time steps	500
Simulation time	250 s



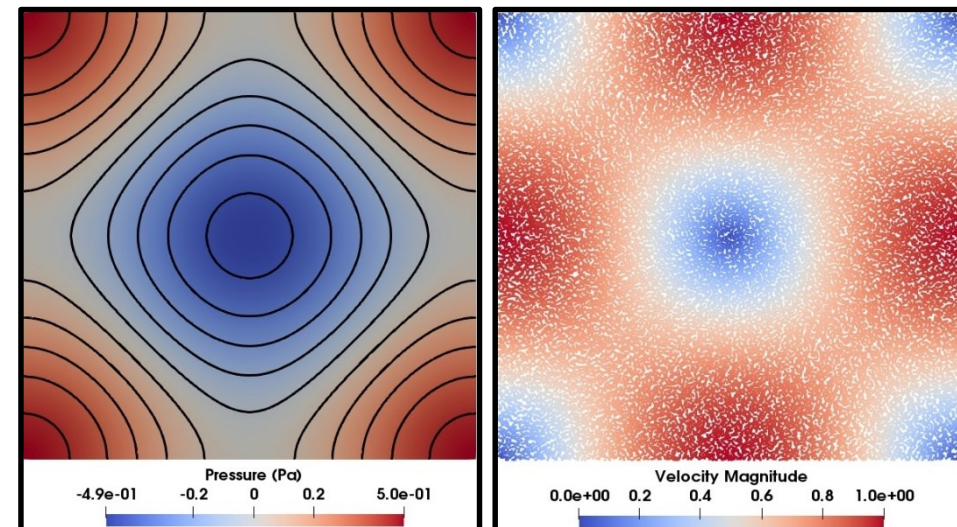


# TAYLOR-GREEN VORTEX

## Taylor-Green vortex decay

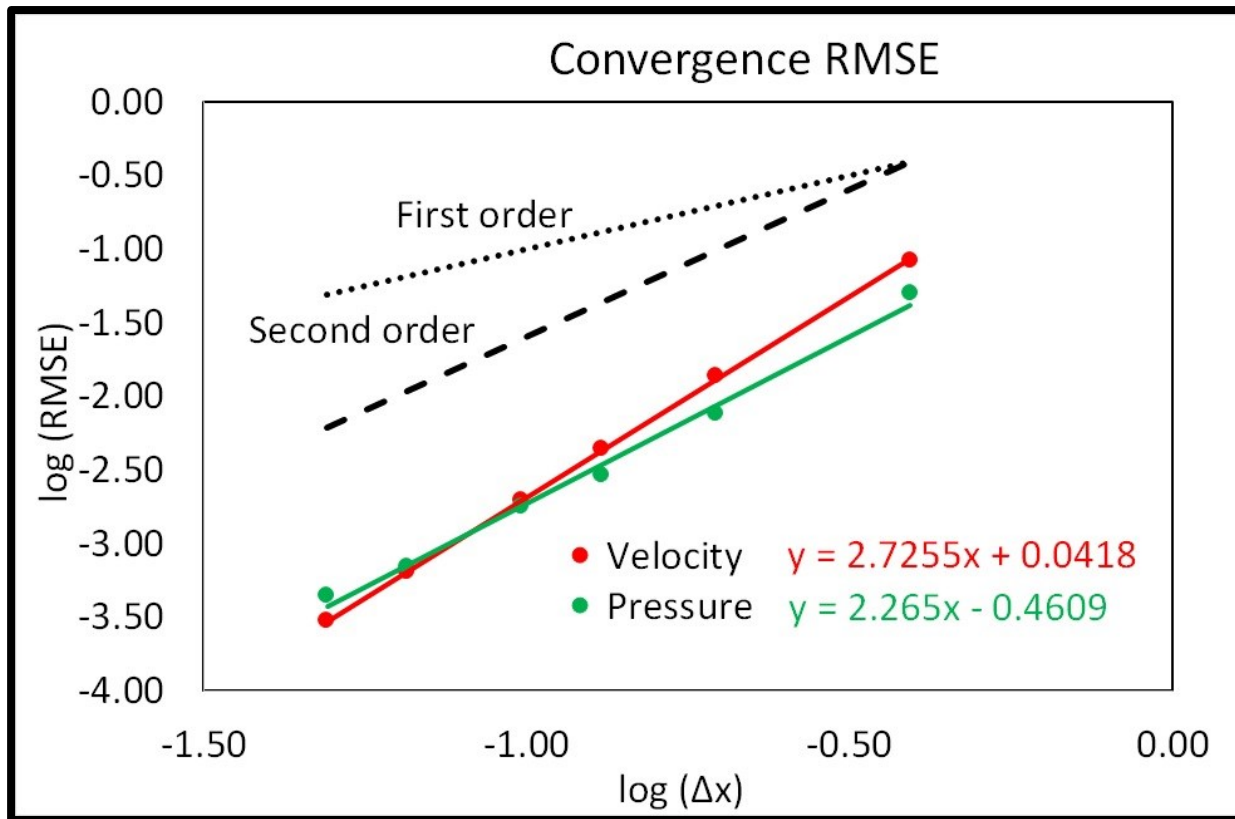


Particulars	
Maximum velocity (m/s)	1
Reynolds number	<b>3140</b>
Domain size (mxm)	$(0, \Pi) \times (0, \Pi)$
Courant number	<b>0.5</b>
Simulation time (s)	10



# CONVERGENCE ANALYSIS

## Taylor-Green vortex decay



Case	$\Delta x(m)$	$\Delta t(s)$	$N \Delta t$
8x8	$\pi/8$	0.2	50
16x16	$\pi/16$	0.1	100
24x24	$\pi/24$	0.0667	150
32x32	$\pi/32$	0.05	200
48x48	$\pi/48$	0.0333	300
64x64	$\pi/64$	0.025	400



# CONVERGENCE ANALYSIS

## Steady-state Taylor Green Vortex

### Mass forces:

$$f_x(x, y) = -2\nu \sin(x) \cos(y)$$

$$f_y(x, y) = 2\nu \cos(x) \sin(y)$$

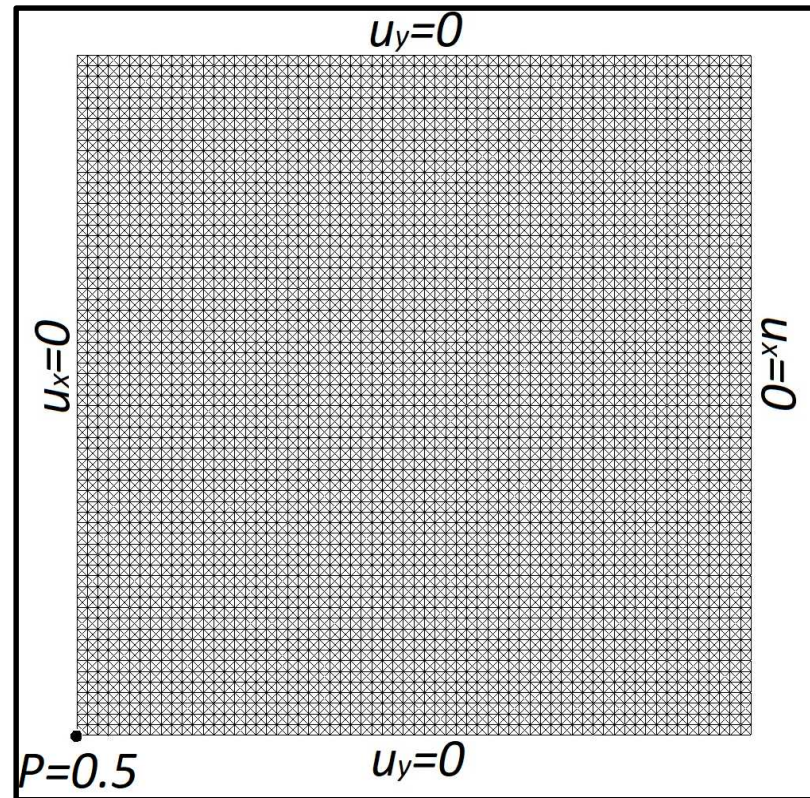
### Analytical solution:

$$u_x(x, y) = -\sin(x) \cos(y)$$

$$u_y(x, y) = \cos(x) \sin(y)$$

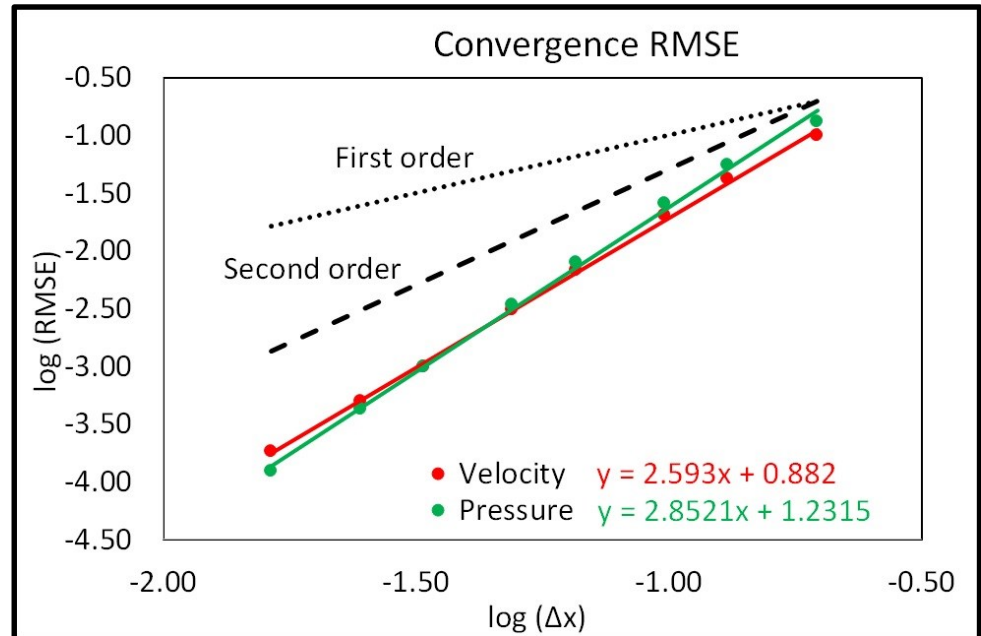
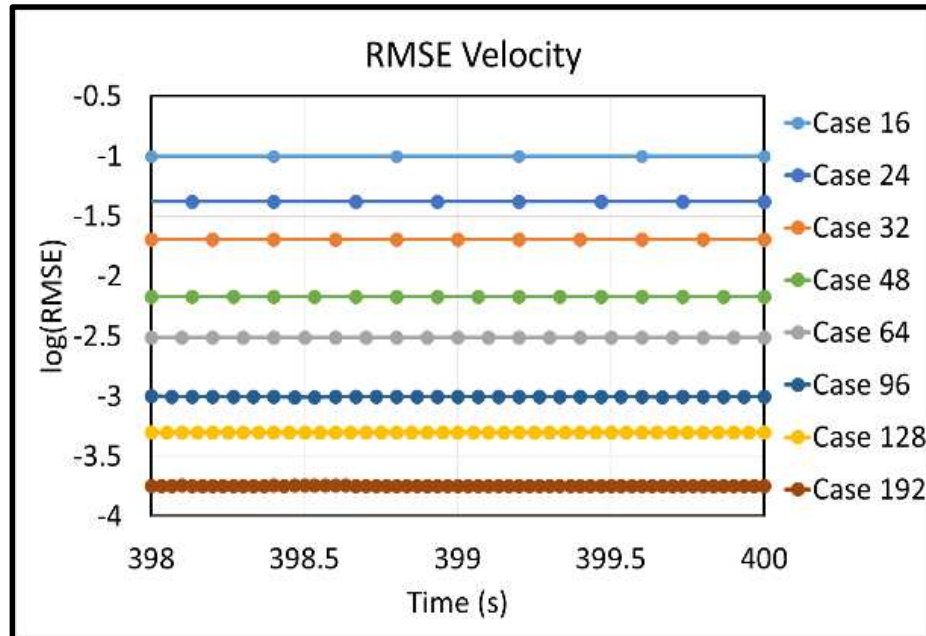
$$P(x, y) = 0.25[\cos(2x) + \cos(2y)]$$

Particulars	
Maximum velocity (m/s)	1
Reynolds number	<b>314</b>
Domain size (mxm)	$(0, \Pi) \times (0, \Pi)$
Courant number	<b>2.04</b>
Simulation time (s)	400



# CONVERGENCE ANALYSIS

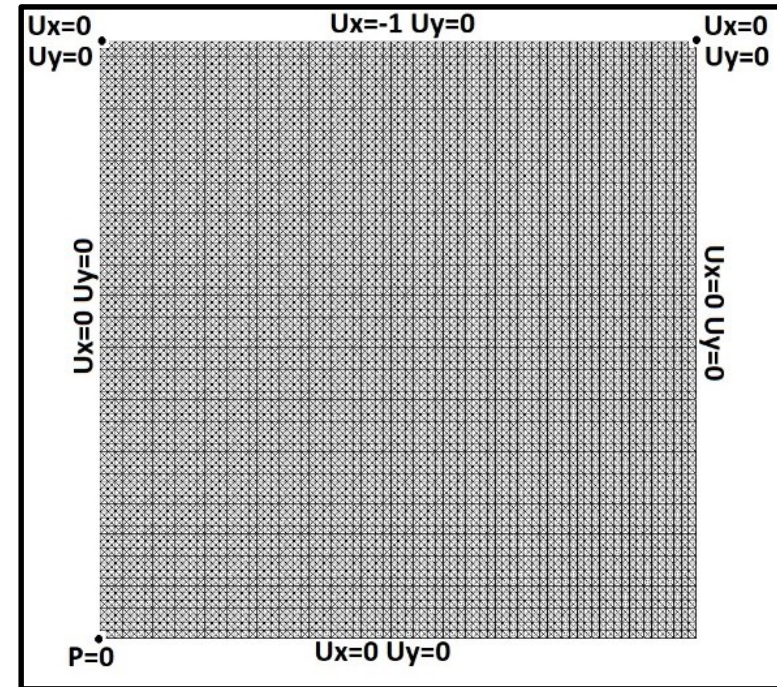
Case	$\Delta x(m)$	$\Delta t(s)$	N $\Delta t$
16x16	$\pi/16$	0.4	1000
24x24	$\pi/24$	0.2667	1500
32x32	$\pi/32$	0.2	2000
48x48	$\pi/48$	0.1333	3000
64x64	$\pi/64$	0.1	4000
96x96	$\pi/96$	0.0667	6000
128x128	$\pi/128$	0.05	8000
192x192	$\pi/192$	0.0333	12000





# LID DRIVEN CAVITY FLOW

Lid velocity $V$	$-1m/s$
Reynolds number $Re$	1000
Domain size	$1m \times 1m$
Domain discretization	$80 \times 80$
Number of elements	25600
Number of nodes	12961
Particles per element	3
Mesh size $\Delta x = \Delta y$	0.0125
Time step $\Delta t$	0.1
Courant number	<b>8</b>
Simulation time	$10^5 \Delta t$
Sampling time	$10^2 \Delta t$

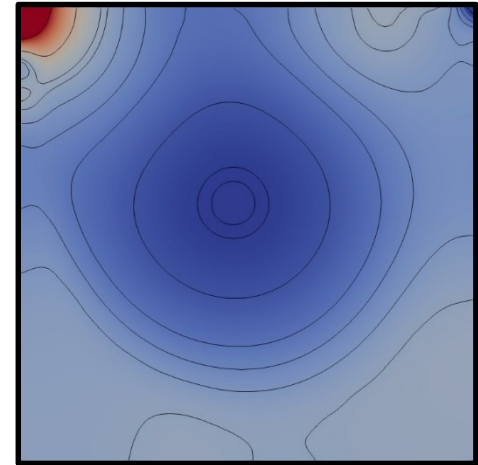
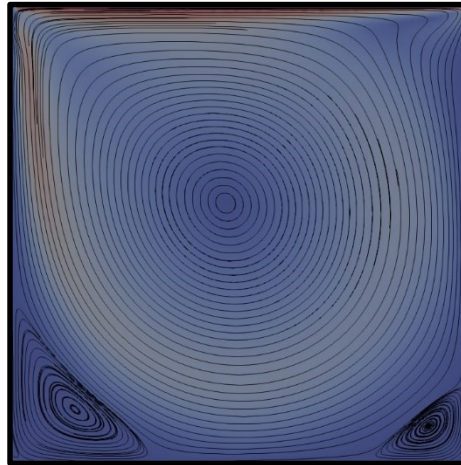


# LID DRIVEN CAVITY FLOW

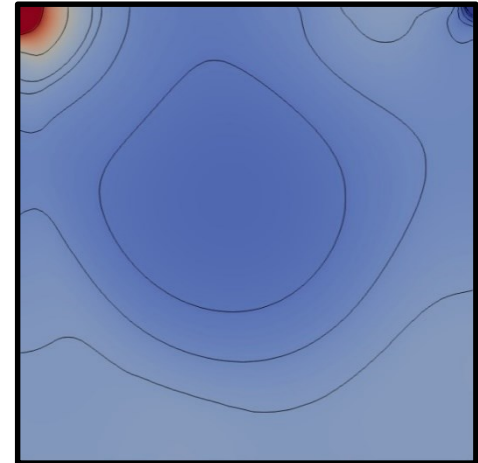
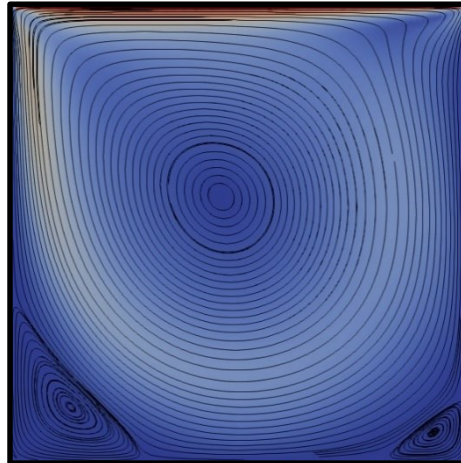
Velocity-streamlines

Pressure

Second order Verlet  
Verlet SL-PFEM



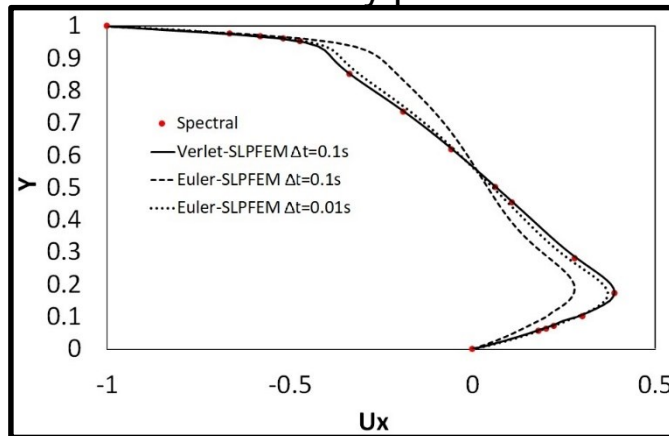
First order in time  
Euler SL-PFEM



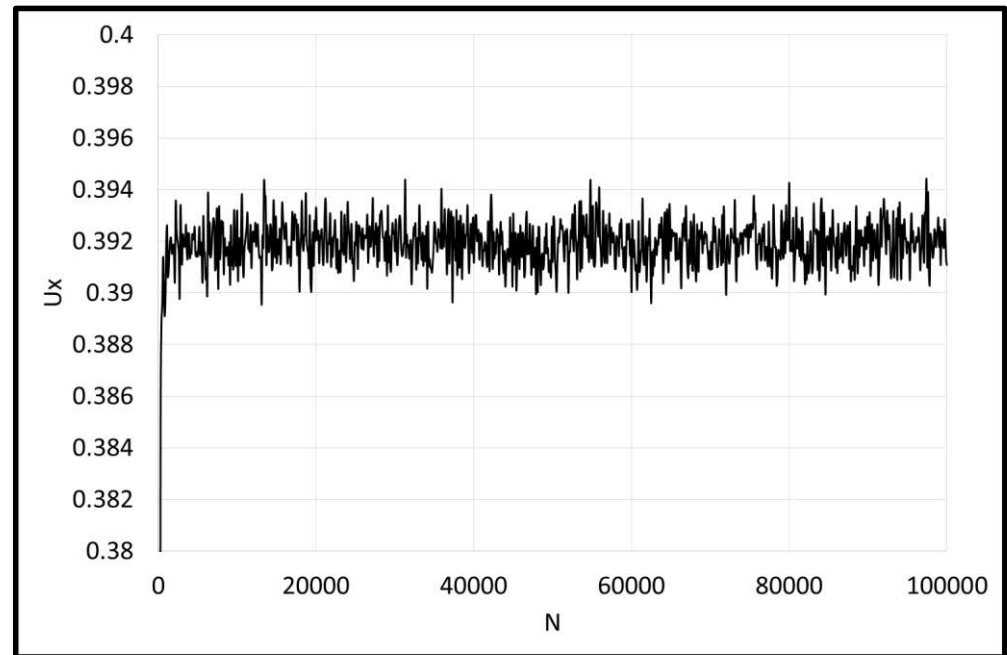
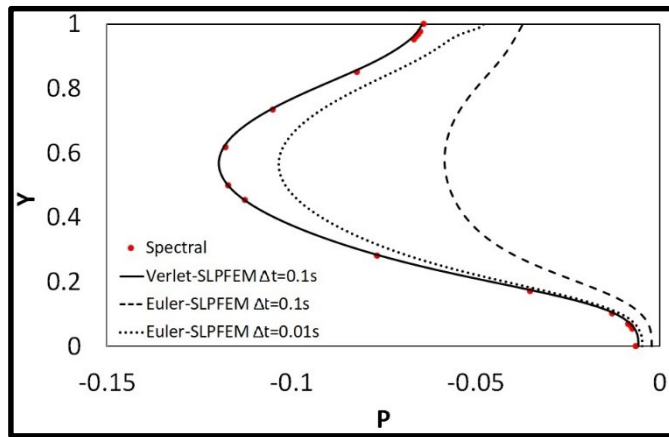
# LID DRIVEN CAVITY FLOW

Horizontal velocity and pressure profiles at the mid-section  $x=0.5$ . Solid line: Second Order Verlet-SLPFEM. Dash line: First Order in time Euler-SLPFEM. Red dots: spectral method (O. Botella and R. Peyret. Benchmark spectral results on the lid-driven cavity flow. Computers & Fluids Vol. 27, No. 4, pp. 421-433, 1998)

Velocity profile



Pressure profile



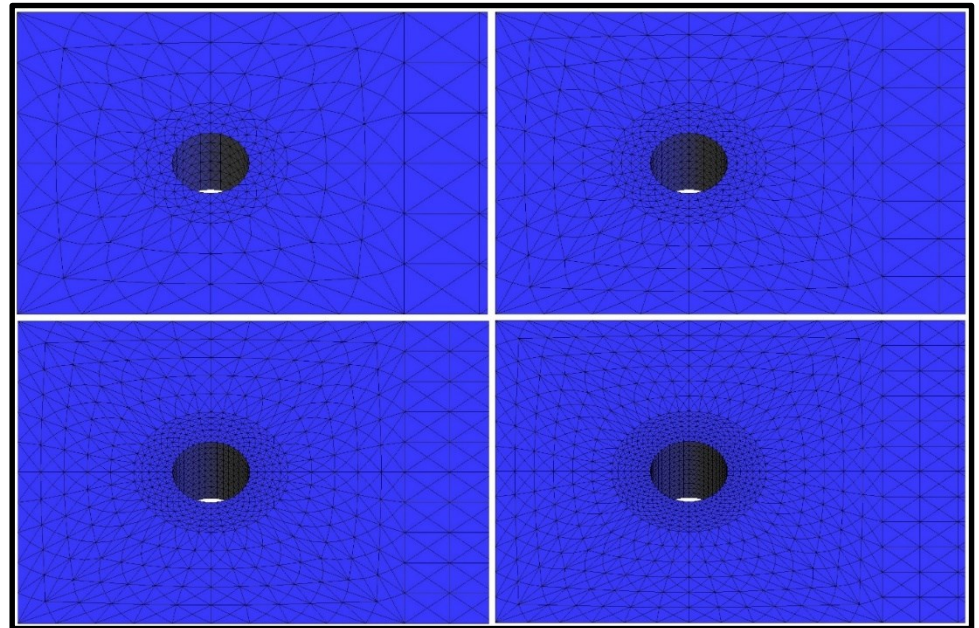
Horizontal velocity evolution at  $(x,y)=(0.5,0.175)$   
for the second-order Verlet -SLPFEM



# 3D FLOW PAST A CYLINDER

Mesh	$\Delta x(\text{m})$	$\Delta t(\text{s})$	N tetras	N Nodes	N $\Delta t$
1	0.1333	0.0667	62208	14487	1500
2	0.1	0.05	147456	33412	2000
3	0.08	0.04	288000	64185	2500
4	0.0667	0.0333	497664	109686	3000

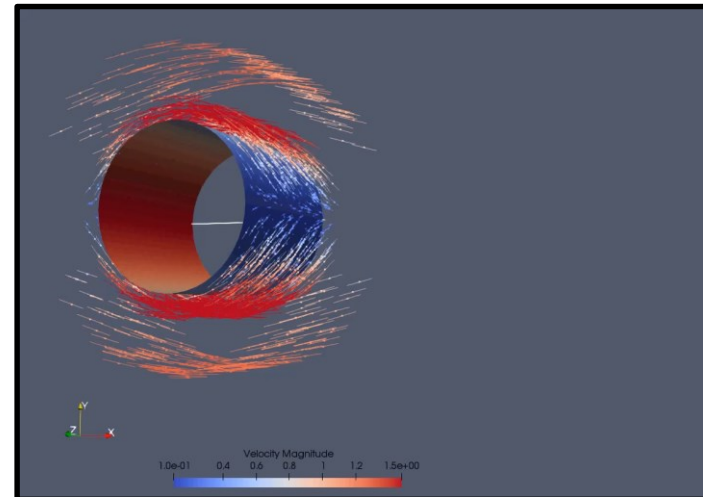
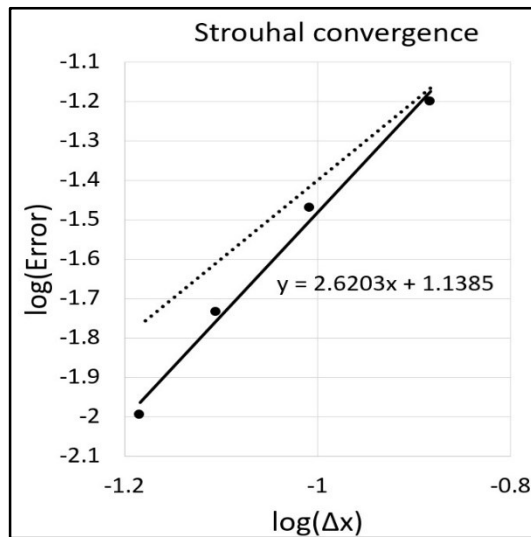
Particulars	
Cylinder diameter (m)	1
Cylinder height (m)	1
Inlet velocity (m/s)	1
Reynolds number	200
Courant number	2



# 3D FLOW PAST A CYLINDER

Case	St	Error St	$C_L$
1	0.133	0.0630	0.177
2	0.162	0.0339	0.255
3	0.178	0.0184	0.246
4	0.186	0.0101	0.258
Ref. [*]	0.196		

CPU time in seconds per time step (mesh 2)			SL-PFEM		FEM
Particles	Move		0.092	0.602	
	Projector	Assembly	0.363		
		Solver	0.147		
Velocity solver			0.125		1.532
Pressure solver			0.339		0.320
Total			1.066		1.852



\* H. Jiang and L. Cheng. Strouhal Reynolds number relationship for flow past a circular cylinder. J. Fluid Mech. (2017), vol. 832, pp. 170-188

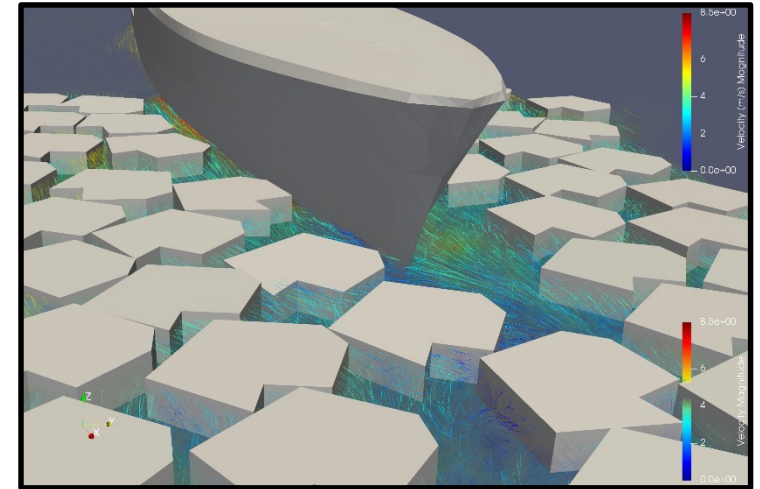


# ONGOING AND FUTURE WORK

# ONGOING AND FUTURE WORK

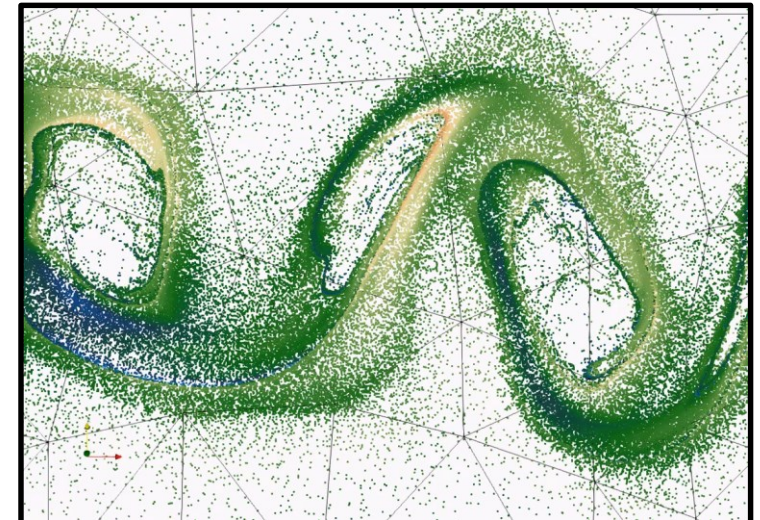
## Ongoing Work

- ✓ FEM Enrichment for fluid interfaces
  - ✓ Fluid solid interaction:  
Advances in the simulation of ship navigation in ice  
(IS-Ships in ice, Room: Runan @ 16:00-17:40h)
- ✓ Free surface problems.



## Future Work

- ✓ Solving submesh scales
  - ✓ Solve particle scale vorticity using a particle based vortex method.
- ✓ Turbulence modelling



# ACKNOWLEDGEMENTS

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